AIAA JOURNAL
Vol. 34, No. 2, February 1996

Iterative Least-Squares Calculation for Modal Eigenvector Sensitivity

Jean-Guy Béliveau*
University of Vermont, Burlington, Vermont 05405-0156
and
Scott Cogan,† Gérard Lallement,‡ and Frederique Ayer§
Université de Franche-Comté, Besançon 25030, France

There are a number of applications where the sensitivity of eigenvectors with respect to physical parameters is desired. We develop an iterative solution scheme for calculating the eigenvector sensitivity in which only the lowest eigencharacteristics are required. It uses a least-squares formulation for the eigenvector sensitivity including the relation from the basic eigenvalue problem and the orthogonality and normality conditions with respect to the mass matrix. The iterative scheme uses the band structure of the stiffness matrix and an efficient use of the Householder transformation to reduce the number of calculations. Since only the lowest eigensolutions are used in the formulation, it is applicable to situations where only a partial eigensolution of the lowest eigenvectors and eigenvalues is available. The eigenvalues are assumed to be distinct and only the first-order variation is calculated. The stiffness matrix is assumed to be nonsingular and a Choleski decomposition of the stiffness matrix is required, but this is the only large matrix that needs to be decomposed. The least-squares solution to the eigenvector sensitivity is shown to reduce to the modal expansion method when appropriate weights are incorporated. From this expression, we show why the modal expansion is not always adequate for eigenvector sensitivity and give a criterion for evaluating this method in a given application.

Nomenclature

A	= matrix used in least-squares representation of eigenvector sensitivity
A_{ij}, B_{ij}	= terms used in modal expansion of eigenvector sensitivity
a	= scalar used in Nelson technique
B, C	= matrices used in Householder modification to iterative procedure
c, c_1, c_2	= matrix of vectors used in kth eigenvector sensitivity
D, d_i	= diagonal matrix and its ith diagonal element
e, f, g, g_1, g_2, h	= vectors used for sensitivity of eigenvectors
K'	= transpose of K
K^{-1}	= inverse of K
K, M	= stiffness and mass matrices, respectively
m	= number of terms in the modal expansion or the normalization term
N_m, M_m	= ratios of terms in modal expansion method
n	= order of matrix system
p	= parameter
U, V	= matrices used in Householder representation of eigensensitivity
w, W	= scalar and diagonal weight of normality and orthogonality conditions
x, y, z	= vectors used to represent parts of eigenvector sensitivity
δ^l	= correction vector of eigenvector sensitivity at l iteration
Φ	= matrix of eigenvectors, i.e., mode shapes
ϕ	= mode shape of vibration
Ω^2	= diagonal matrix of square of eigenvalues, i.e., resonant frequencies
ω, ω^2	= natural frequency of vibration and elements of Ω^2

Received Jan. 15, 1994; revision received Sept. 1, 1995; accepted for publilcation Sept. 5, 1995.

*Professor, Department of Civil and Environmental Engineering. Member AIAA.
†Research Associate, Laboratoire de Mécanique Appliquée R. Chaléat.
‡Professeur, Laboratoire de Mécanique Appliquée R. Chaléat.
§Graduate Student, Laboratoire de Mécanique Appliquée R. Chaléat.

Introduction

WITTRICK[1] gave an early formulation of the sensitivity of the buckling load and the resonant frequencies of structural systems that were a function only of the particular eigenvalue and its eigenvector. Fox and Kapoor[2] gave an algebraic method to calculate the eigenvector sensitivity incorporating the normality constraint for symmetric matrix systems. Plaut and Huseyin[3] extended it to non-self-adjoint systems, and Wang[4] accelerated the convergence by adding a static solution. Rudisill[5] later developed expressions for higher order derivatives. These algebraic techniques require all of the eigenvalues and eigenvectors of the system, however, rendering it prohibitive in models with thousands of degrees of freedom where partial eigenvalue solvers are used to get the lowest frequencies and mode shapes of vibration. For large matrix systems, it is expensive to obtain all of the eigensolutions.

Thus techniques that do not require all of the eigenvalues and eigenvectors are preferable. Fox and Kapoor[2] also developed such a technique that is iterative in nature and that included the normality condition. The eigenvector derivative for the first eigenvalue depended only on the first eigenvalue and its eigenvector. Though it converged for the first eigenvalue, it did not always converge for the higher eigenvalues.

Rudisill and Chu[6] incorporated orthogonality conditions in this iterative scheme and observed that the iterative solution converged even for higher modal vectors when orthogonality conditions were incorporated. Andrew[7] showed convergence of this iterative method under fairly general requirements. This iterative technique, however, yielded a matrix that altered the band structure of the original mass and stiffness matrices.

Nelson[8] developed an exact technique that maintains the symmetry and band structure of the original system but required a decomposition at each eigenfrequency for which sensitivity is required. It does not incorporate the orthogonality conditions, however. Murthy and Haftka[9] have demonstrated that a normalization in which one element has a fixed amplitude is better than a normalization based on the length of the vector, particularly for complex eigenvectors. Cardani and Mantegazza[10] extended Nelson's technique to the quadratic eigenvalue problem for flutter and divergence eigenproblems. Ojalvo[11] proposed both a direct and an iterative version of Nelson's method and proposed an extension of it for problems with repeated eigenvalues. As in Nelson's method, a Choleski decomposition of each $n \times n$ matrix $K - \omega^2 M$ is needed for all ω^2 for which sensitivity is required.

Using a least-squares formulation we demonstrate why the modal expansion method is in general not adequate for eigenvector sensitivity calculations. We give a criterion with which the expansion can be evaluated. Next, we extend the iterative formulation of Rudisill and Chu[6] in a least-squares context weighing both normality and orthogonality conditions. The resulting equation is solved in an iterative fashion. The mass and stiffness matrices are assumed to be symmetric, the stiffness matrix is further assumed to be nonsingular, and the eigenvalues are assumed to be distinct. The use of Householder's transformation makes this iterative approach efficient. Thus, only one large matrix, the stiffness matrix, needs to be decomposed. The sensitivity for the kth eigenvector is expressed in terms of the k-lowest eigencharacteristics, and there is no need to incorporate the higher modes. Thus, it is applicable to large matrix systems with partial eigensolutions. Furthermore, we give a criterion on the minimum weight to use for the orthogonality and normality conditions to assure convergence of the method.

Eigenvector Sensitivity

The eigenvalue problem is of the form

$$[K - \omega^2 M]\phi = 0 \tag{1}$$

The stiffness K and mass M matrices are $n \times n$ matrices assumed to be symmetric:

$$M' = M \qquad K' = K \tag{2}$$

where the superscript $'$ represents the transpose of the matrix.

In addition, the stiffness matrix is assumed to be positive definite, and it is assumed that the eigenvalues ω^2, which are the squares of the resonant natural frequencies, and corresponding eigenvectors ϕ, which are the mode shapes of vibration, are distinct; i.e., the roots are not repeated.

Taking the partial derivative of the previous equation and applying a premultiplication by the transpose of the eigenvector yield the equation for the sensitivity of the eigenvalue

$$\frac{\partial \omega^2}{\partial p} = \frac{\phi'[(\partial K/\partial p) - \omega^2(\partial M/\partial p)]\phi}{\phi' M \phi} = \phi'\left[\frac{\partial K}{\partial p} - \omega^2\frac{\partial M}{\partial p}\right]\phi \tag{3}$$

in which the following normality constraint has been used:

$$\phi' M \phi = 1 \tag{4}$$

The first-order derivative of the eigenvalue problem, Eq. (1), gives the eigenvector sensitivity

$$[K - \omega^2 M]\frac{\partial \phi}{\partial p} = f = -\left(\frac{\partial K}{\partial p} - \omega^2\frac{\partial M}{\partial p} - \frac{\partial \omega^2}{\partial p}M\right)\phi \tag{5}$$

The matrix on the left-hand side is singular with rank equal to $n-1$. The normality condition gives us the additional condition required to make the sensitivity calculation unique. Applying chain rule differentiation to Eq. (4) gives

$$\phi' M\frac{\partial \phi}{\partial p} = -\frac{1}{2}\phi'\frac{\partial M}{\partial p}\phi \tag{6}$$

Equations (5) and (6) were formulated by Fox and Kapoor.[2] It should be noted that this method does not include the orthogonality conditions, which are automatically satisfied by the exact eigenvectors.

Nelson Method

Nelson[8] gave an exact method for solving the eigenvector sensitivity based on expanding the sensitivity as a sum of a multiple of the particular eigenvector being considered plus an additional vector which is M orthogonal to the vector under consideration,

$$\frac{\partial \phi}{\partial p} = a\phi + z \tag{7}$$

The mth term of the vector z, normally the term with the largest eigenvector amplitude, is set equal to zero to avoid singular sensitivity matrices:

$$z_m = 0 \tag{8}$$

The original matrix system for the eigenvector sensitivity can then be modified to be a nonsingular matrix equation that replaces the mth equation, which was a linear combination of the other $n-1$ equations, thus rendering the original system singular. The new matrix is of the form

$$\begin{bmatrix} K_{11} - \omega^2 M_{11} & 0 & K_{12} - \omega^2 M_{12} \\ 0 & w & 0 \\ K_{21} - \omega^2 M_{21} & 0 & K_{22} - \omega^2 M_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_m \\ z_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ f_2 \end{bmatrix} \tag{9}$$

where w is an appropriate weight to scale the normalization equation, $z_m = 0$, to the same order as the other equations; i.e., the matrix $K - \omega^2 M$ is modified to have a zero column and row except for the mth diagonal element that forces the mth component of the vector z to be zero. The matrix system is banded but nevertheless necessitates a considerable amount of computer effort when sensitivity of more than one eigenvector is needed, since it must be decomposed for each resonant frequency for which the eigenvector sensitivity is of interest. The parameter a in Eq. (7) is obtained from the normality condition, Eq. (4):

$$a = -\frac{1}{2}\phi'\frac{\partial M}{\partial p}\phi - \phi' M z \tag{10}$$

Modal Expansion Method

Since the eigenvectors are independent and span the space, the eigenvector sensitivity of the kth eigenvector can be expanded as a sum of the eigenvectors, i.e.,

$$\frac{\partial \phi_k}{\partial p} = \sum_{j=1}^{n} A_{kj}\phi_j \tag{11}$$

which, when substituted into Eq. (5) and premultiplied by the transpose of the jth modal shape, gives for j not equal to k (see Fox and Kapoor[2])

$$A_{kj} = \frac{\phi_j'\left[(\partial K/\partial p) - \omega_k^2(\partial M/\partial p)\right]\phi_k}{\omega_k^2 - \omega_j^2} \tag{12}$$

The term $j = k$ is obtained from the sensitivity of the normality condition, Eq. (6):

$$A_{kk} = -\frac{1}{2}\phi_k'\frac{\partial M}{\partial p}\phi_k \tag{13}$$

Accelerated Modal Method

To accelerate convergence, the eigenvector sensitivity can be written as a sum of a static component x for $Kx = f$, and a modal expansion, which leads to[4]

$$\frac{\partial \phi_k}{\partial p} = x_k + \sum_{j=1}^{n} B_{kj}\phi_j \tag{14}$$

in which the static solution satisfies

$$K x_k = f_k \tag{15}$$

The term B_{kj} for k not equal to j is given by

$$B_{kj} = \left(\frac{\omega_k^2}{\omega_j^2}\right)\frac{\phi_j'\left[(\partial K/\partial p) - \omega_k^2(\partial M/\partial p)\right]\phi_k}{\omega_k^2 - \omega_j^2} \tag{16}$$

The term with $j = k$ is again obtained from the sensitivity of the normality condition, Eq. (6), and is the same as the earlier expression, Eq. (13). For k less than j, the coefficients B_{kj} are multiplied by a ratio less than 1, thus accelerating convergence, leading to better results with fewer terms than the procedure without the static solution. The stiffness matrix K needs to be decomposed for the static solution $Kx_k = f_k$ only once, but the corresponding static solutions need to be solved for each eigenvector for which sensitivity is sought. Wang[4] further extended this method to accelerate convergence by introducing a residual static mode where contributions of the lower modes are subtracted from the static mode.

Orthogonality Conditions

The orthogonality of the eigenvector with the other $k-1$ lower eigenvectors when taken with respect to the mass matrix can also be integrated with the original system in addition to the normality constraint

$$\phi_i' M\phi = 0 \qquad \text{for} \quad i = 1, 2, \ldots, k-1 \tag{17}$$

The following set of equations are obtained by differentiating the normality, Eq. (4) and orthogonality conditions, Eq. (17),

$$c'M\frac{\partial\phi}{\partial p} = g = \begin{bmatrix} -\phi_1'\dfrac{\partial M}{\partial p} - \dfrac{\partial\phi_1'}{\partial p}M \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ -\phi_{k-1}'\dfrac{\partial M}{\partial p} - \dfrac{\partial\phi_{k-1}'}{\partial p}M \\ -\dfrac{1}{2}\phi'\dfrac{\partial M}{\partial p} \end{bmatrix}\phi \tag{18}$$

in which c' is the transpose of a rectangular matrix c, which is composed of the k eigenvectors and has dimensions $n \times k$

$$c = [\phi_1 \cdots \phi_{k-1}\,\phi] \tag{19}$$

The combined solution to Eqs. (5) and (18) may be written in a least-squares sense as

$$[A'A]\frac{\partial\phi}{\partial p} = A'e \tag{20}$$

in which

$$A = \begin{bmatrix} K - \omega^2 M \\ U' \end{bmatrix} \qquad e = \begin{bmatrix} f \\ g \end{bmatrix} \tag{21}$$

where U is given by the following:

$$U = Mc \tag{22}$$

The symmetric system, Eq. (20), must be solved for each eigenvalue considered. Also, the band structure of the original matrix system is destroyed by the least-squares formulation. As will be demonstrated later, however, the Householder decomposition will be used to avoid the decomposing of the matrix $A'A$.

Modal Least-Squares Method

Since K and M are symmetric, Eqs. (5) and (18) can be added differently and still lead to a symmetric matrix. When the derivatives of the normality and orthogonality conditions, Eq. (18), are weighted, the sum of the two equations appropriately rendered symmetric may be written as

$$[K - \omega^2 M + UWU']\frac{\partial\phi}{\partial p} = h \tag{23}$$

in which U is given by Eq. (22) and W is a diagonal weighing matrix. The vector h is given by

$$h = f + UWg \tag{24}$$

in which f is given in Eq. (5) and g is given in Eq. (18). The stiffness and mass matrices can be reformulated in terms of the matrix of eigenvectors Φ due to the orthogonality conditions with respect to the stiffness and mass matrices

$$\begin{aligned} \Phi' M\Phi &= I & \Phi' K\Phi &= \Omega^2 \\ M &= \Phi^{-T}\Phi^{-1} & K &= \Phi^{-T}\Omega^2\Phi^{-1} \end{aligned} \tag{25}$$

where $-T$ is the inverse of the transpose matrix. The orthogonality and normality conditions may also be written in terms of the matrix of eigenvectors Φ and a diagonal matrix, which is a function of the weights,

$$\begin{aligned} UWU' &= McWc'M = \Phi^{-T}\Phi^{-1}cWc'\Phi^{-T}\Phi^{-1} \\ &= \Phi^{-T}\begin{bmatrix} I \\ 0 \end{bmatrix} W[I \quad 0]\Phi^{-1} = \Phi^{-T}\begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}\Phi^{-1} \end{aligned} \tag{26}$$

in which the following fact has been used:

$$\Phi^{-1}c = \Phi^{-1}[\phi_1 \quad \phi_2 \quad \cdots \quad \phi_k] = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{27}$$

Substitution of Eqs. (25) and (26) into Eq. (23) yields

$$\Phi^{-T}D\Phi^{-1}\frac{\partial\phi}{\partial p} = h \tag{28}$$

in which D is a diagonal matrix with diagonal elements given as

$$d_{ii} = \begin{bmatrix} \omega_i^2 - \omega^2 + W_i & \text{for} \quad i = 1, 2, \ldots, k \\ \omega_i^2 - \omega^2 & \text{for} \quad i = k+1, \ldots, n \end{bmatrix} \tag{29}$$

The solution to the preceding for a diagonal weighing matrix W may be written down in modal form as

$$\frac{\partial\phi}{\partial p} = \Phi D^{-1}\Phi' h = \left[\sum_{j=1}^{n}\frac{1}{d_{jj}}\phi_j\phi_j'\right]h = \sum_{j=1}^{n}\frac{\phi_j' h}{d_{jj}}\phi_j \tag{30}$$

Modal Expansion Formulation

Substituting the value for h, f, and U, Eq. (30) becomes

$$\frac{\partial\phi}{\partial p} = \left[\sum_{j=1}^{n}\frac{\phi_j\phi_j'}{d_{jj}}\right] \times \left\{McWg + \left[-\frac{\partial K}{\partial p} + \omega^2\frac{\partial M}{\partial p} + \frac{\partial\omega^2}{\partial p}M\right]\Phi\right\} \tag{31}$$

When the orthogonality conditions are omitted, i.e., their weight is set equal to zero, and the normality condition is weighted by one, there results for $\phi = \phi_k$

$$\frac{\partial\phi_k}{\partial p} = \sum_{\substack{j=1 \\ j\neq i}}^{n}\left[\frac{\phi_j'[(\partial K/\partial p) - \omega_k^2(\partial M/\partial p)]\phi_k}{\omega_k^2 - \omega_j^2}\right]\phi_j - \frac{1}{2}\left[\phi_k'\frac{\partial M}{\partial p}\phi_k\right]\phi_k \tag{32}$$

in which

$$d_{ii} = \begin{bmatrix} \omega_i^2 - \omega_k^2 + W_i = 1 & \text{for} \quad i = k \\ \omega_i^2 - \omega_k^2 & \text{for} \quad i < k \end{bmatrix} \tag{33}$$

This is exactly the same result presented earlier in Eqs. (11–13) for the modal expansion technique. Therefore, the modal expansion technique is a subset of the least-squares technique in which the orthogonality constraints are not included and the normality constrained is weighted by the value 1.

The advantage of the least-squares technique is that it integrates the orthogonality conditions automatically into the procedure. This is particularly useful for large systems for which a subset of the eigenvectors are available rather than the whole set and a truncated modal expansion must be used in evaluating the eigenvector sensitivity. Although the theoretical eigenvectors automatically satisfy the orthogonality conditions, the vectors used in an iterative process do not. Imposing the orthogonality conditions in addition to the normality conditions forces the vectors to satisfy both normality and orthogonality.

Truncated Modal Expansion

Equation (30) is exact when all of the terms in the summation are taken but is approximate when the number of modes used in the expansion, m, is less than n, the order of the matrix system. Unfortunately, the modal method described does not imply that the terms associated with the higher order mode shapes are smaller than those associated with the lower mode shapes. This can be easily shown by comparing the norm of the added term from the mth eigenvector to that of the value obtained from the $m-1$ eigenvectors. From Eq. (30), the length of the added term divided by the length of the vector with the m-lower terms is

$$N_m = \frac{|(\phi_m' h/d_{mm})\phi_m|}{\left|\sum_{j=1}^{m-1}(\phi_j' h/d_{jj})\phi_j\right|} \tag{34}$$

The denominator is zero when the vector h is orthogonal to the $m-1$ lowest eigenvectors, thus making the mth term arbitrarily large. Since h is in general arbitrary, the modal representation of the eigenvector sensitivity is not to be recommended since a term left out could be the dominant term, i.e., when h is a linear combination of the mode shapes left out in the expansion.

An alternate ratio can be used to ascertain whether or not the modal expansion is adequate or not. In particular, h can be expanded as a function of the eigenvectors

$$h = \sum_{j=1}^{m} a_j\phi_j + \sum_{j=m+1}^{n} a_j\phi_j \tag{35}$$

The jth coefficient may be obtained by a Gram–Schmidt procedure with a premultiplication by the jth eigenvector and either the mass M or stiffness K matrix, with respect to which the eigenvectors are orthogonal,

$$\begin{aligned} a_j &= \frac{\phi_j' M h}{\phi_j' M \phi_j} = \phi_j' M h \quad (M\text{-norm}) \\ a_j &= \frac{\phi_j' K h}{\phi_j' K \phi_j} = \frac{\phi_j' K h}{\omega_j^2} \quad (K\text{-norm}) \end{aligned} \tag{36}$$

The approximation is then valid if the length of the terms left out is small with respect to the norm of the vector h,

$$M_m = \frac{\left|h - \sum_{j=1}^{m} a_j\phi_j\right|}{|h|} \tag{37}$$

The length M_m, which can also be defined with respect to the identity matrix or the mass or stiffness matrix, can be as small as required but would necessitate additional terms in the expansion. For accurate determination by the modal expansion technique, the smaller M_m is, the better the partial modal expansion approximation should be. Were h to be orthogonal to the lowest $k-1$ eigenvectors, M_n would be equal to 1.

Iterative Least-Squares Method

The least-squares formulation can also be written in a iterative manner (see Fox and Kapoor[2] and Rudisill and Chu[6]). From Eq. (23),

$$[K + UWU']\frac{\partial\phi}{\partial p} = h + \omega^2 M \frac{\partial\phi}{\partial p} \tag{38}$$

The partial derivative satisfies the equation exactly, whereas for a vector y^l, which is an approximation to the sensitivity, it satisfies the iterative relation

$$[K + UWU']y^l = h + \omega^2 M y^{l-1} \tag{39}$$

In the iterative process, the initial derivative on the right-hand side is set equal to zero and then updated at each iteration until convergence. For zero weight of the normality and orthogonality conditions, $W=0$, and for a static approximation, $\omega=0$, this reduces to the static result used to accelerate convergence in the modal method[4]:

$$K y^l = f \tag{40}$$

Householder Adaptation

Although the band structure of the stiffness matrix is destroyed in Eq. (39), the Householder transformation can be used to solve the preceding system. From the Appendix, the Householder transformation[12] gives

$$[K + UWU']^{-1} = K^{-1}[K - UB^{-1}U']K^{-1} \tag{41}$$

in which a much smaller matrix B of dimension $k \times k$ needs to be decomposed, in addition to K, where k is the order of the eigenvalue, usually a small number, since low frequencies are of interest. The term B is given by (see the Appendix)

$$B = W^{-1} + U'K^{-1}U \tag{42}$$

A Choleski or LDL' (lower triangular-diagonal-lower triangular transpose) decomposition of K is used rather than its inverse to maintain the band nature of the matrix operations. With this information, the iterative least-squares techniques can be numerically efficient provided not too many iterations are required.

Convergence of Iterative Method

Andrew[7] demonstrated that the technique of Rudisill and Chu[6] converged for the sensitivity of the eigenvector. The procedure developed here also converges. Let δ^l be the error between the exact and approximate y^l value of the eigenvector sensitivity and δ^{l-1} be the error for the sensitivity at the previous iteration,

$$\delta^l = \frac{\partial\phi}{\partial p} - y^l \qquad \delta^{l-1} = \frac{\partial\phi}{\partial p} - y^{l-1} \tag{43}$$

Subtraction of Eq. (39) from Eq. (38) and substitution of Eqs. (25) and (26) yield an expression for the error at the lth iteration,

$$[K + UWU']\delta^l = \Phi^{-T}\left[\Omega^2 + \begin{matrix} W & 0 \\ 0 & 0 \end{matrix}\right]\Phi^{-1}\delta^l = \omega^2 M \delta^{l-1} \tag{44}$$

The error at the previous iteration, δ^{l-1}, may be written as a linear sum of the linearly independent eigenvectors:

$$\delta^{l-1} = \sum_{i=1}^{n} a_i\phi_i \tag{45}$$

The error at the lth iteration δ^l is expressible in terms of the modal matrix Φ and the elements of a diagonal matrix D, from Eqs. (44) and (45),

$$\begin{aligned} \delta^l &= \omega^2 \Phi D^{-1} \Phi' M \delta^{l-1} \\ &= \omega^2 \sum_{i=1}^{n} \Phi D^{-1} \Phi' M a_i \phi_i \\ &= \omega^2 \sum_{i=1}^{n} \frac{a_i\phi_i}{d_{ii}} \end{aligned} \tag{46}$$

in which the normality and orthogonality conditions with respect to the mass matrix for the lower modes have been used and the elements of the D matrix are

$$d_{ii} = \begin{vmatrix} \omega_i^2 + W_i & \text{for} \quad i = 1, 2, \ldots, k \\ \omega_i^2 & \text{for} \quad i = k+1, k+2, \ldots, n \end{vmatrix} \tag{47}$$

By performing j-sequential iterations, the error at the jth iteration is expressible as

$$[\delta^l]^j = \sum_{i=1}^{n} \left[\frac{\omega^2}{d_{ii}}\right]^j a_i \phi_i \tag{48}$$

This error can be made as small as required provided the numerator of each term is less than the denominator,

$$\frac{\omega^2}{d_{ii}} = \frac{\omega^2}{\omega_i^2 + W_i} < 1 \qquad \text{for} \quad i = 1, 2, \ldots, k$$

$$= \frac{\omega^2}{\omega_i^2} < 1 \qquad \text{for} \quad i = k+1, k+2, \ldots, n \tag{49}$$

The second of these is automatically satisfied, since the frequencies are ranked in increasing order and the first requirement gives the result

$$W_i > \omega^2 - \omega_i^2 \qquad \text{for} \quad i = 1, 2, \ldots, k \tag{50}$$

The largest positive right-hand side for the terms $i = 1, 2, \ldots, k$ is the difference between the kth eigenvalue and the first eigenvalue. If the weights are all set equal to one value, W, it should be at least larger than the difference between the kth and the lowest eigenvalue:

$$W > \omega_k^2 - \omega_1^2 \qquad \text{for} \quad W_1 = W_2 = \cdots = W_k = W \tag{51}$$

Thus convergence is assured provided the weights used in the iterative scheme are larger than the values given next,

$$\begin{array}{ll} W > 0 & \text{for} \quad \omega_1 \\ W > \omega_2^2 - \omega_1^2 & \text{for} \quad \omega_2 \\ \cdots\cdots\cdots\cdots & \\ W > \omega_k^2 - \omega_1^2 & \text{for} \quad \omega_k \end{array} \tag{52}$$

Although the weight W needs to be larger than this value to assure convergence, it is probably not a good idea to set W too large, since the rank of the matrix U that incorporates the normality and orthogonality conditions is k, the order of the eigenvalue, which is much less than n the order of the stiffness matrix. The $n \times n$ system $K + UWU'$ would be nearly singular should the orthogonality and orthonormality conditions be weighted too highly.

Normality Condition for Constant Element

If the normalization is such that the mth element of the eigenvector is to be constant, the normality condition is altered to be

$$\frac{\partial \phi_{mk}}{\partial p} = 0 \tag{53}$$

The normality and orthogonality conditions can be formulated as before but with g and c modified to be

$$U_1' \frac{\partial \phi}{\partial p} = g_1 = - \begin{bmatrix} \phi_1' \dfrac{\partial M}{\partial p} + \dfrac{\partial \phi_1'}{\partial p} M \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \phi_{k-1}' \dfrac{\partial M}{\partial p} + \dfrac{\partial \phi_{k-1}'}{\partial p} M \\ 0 \end{bmatrix} \phi \tag{54}$$

in which U_1 is defined as in Eq. (22), except for the last column, which is

$$u_k = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix} \tag{55}$$

where the number 1 appears in the mth line.

Orthogonality with Respect to the Stiffness Matrix

The conditions that the vectors be orthogonal with respect to the stiffness matrix as well as the mass matrix can also be integrated into the least-squares formulation:

$$\phi_i' K \phi = 0 \qquad \text{for} \quad i = 1, 2, \ldots, k-1 \tag{56}$$

which yields the following additional $k - 1$ equations when differentiated:

$$c_2' K \frac{\partial \phi}{\partial p} = g_2 = - \begin{bmatrix} \phi_1' \dfrac{\partial K}{\partial p} + \dfrac{\partial \phi_1'}{\partial p} K \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \phi_{k-1}' \dfrac{\partial K}{\partial p} + \dfrac{\partial \phi_{k-1}'}{\partial p} K \end{bmatrix} \phi \tag{57}$$

in which g_2 has one less row than g_1 and c_2 has one less column than c or c_1:

$$c_2 = [\phi_1 \cdots \phi_{k-1}] \tag{58}$$

These additional orthogonality conditions can be readily weighted and added to the earlier conditions with respect to the mass matrix in the form

$$[K + UW_1U' + VW_2V'] \frac{\partial \phi}{\partial p} = h_2 + \omega^2 M \frac{\partial \phi}{\partial p} \tag{59}$$

in which

$$h_2 = f + UW_1g_1 + VW_2g_2$$
$$U = Mc_1 \tag{60}$$
$$V = Kc_2$$

The solution of Eq. (59) requires two sequential applications of the Householder transformation, as shown in the Appendix. The only large matrix that needs to be decomposed is the $n \times n$ stiffness matrix K, which is symmetric and positive definite. Furthermore, for a given eigenvalue, it needs to be decomposed only once no matter how many eigenvector sensitivities are required.

Optimum Iterative Least Squares

Equation (59) can be used for the orthogonality and normality concepts for which the second Householder transformation dealing with normality is a scalar. The first decomposition involving the orthogonality constraints need only be performed once for each vector. Thus Eq. (38) may be written as

$$[K + UWU' + vwv'] \frac{\partial \phi}{\partial p} = h_2 + \omega^2 M \frac{\partial \phi}{\partial p} \tag{61}$$

in which U has one fewer column,

$$h_2 = f + UWg_1 + vwg_2$$
$$U = M[\phi_1 \quad \phi_2 \quad \cdots \quad \phi_{k-1}] \tag{62}$$
$$v = M\phi_k$$

The Householder decomposition of the matrix in Eq. (61), which simplifies to a scalar form, is also given in the Appendix.

Examples

Two examples were programmed in MATLAB to compare eigenvector sensitivities of the iterative least-squares technique with those from a modal expansion from Fox and Kapoor.[2] Errors were calculated relative to Nelson's technique as a percentage by using the length of the error vector obtained by subtracting the vector of the sensitivity calculated less than that given by Nelson's method that was then divided by the length of Nelson's sensitivity vector. Initial sensitivity in the iterative least-squares technique was assumed to be the zero vector.

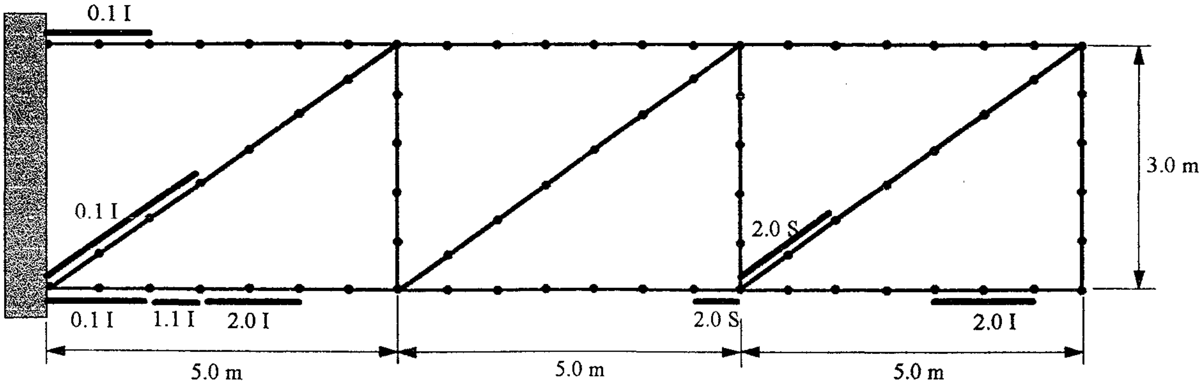

Fig. 1 GARTEUR2 structure.

Table 1 Thirty-two DOF with five modes ($W = 2\omega^2$)

Method	Nelson	Iterative	Modal
flops	65891	581122	14041
CPU(s)	0.14	0.95	0.01
method	least	squares	modal
mode	no. iter.	error, %	error, %
1	9	1.50*E*-5	1.47
2	15	5.62*E*-5	2.95
3	20	1.02*E*-4	4.52
4	24	2.55*E*-4	6.48
5	29	3.89*E*-4	10.56

Table 2 Parameters for GARTEUR2

Young's modulus	$E = 0.75 \times 10^{11}$ Pa
Mass density	$\rho = 2800$ kg m^{-3}
Moment of inertia	$I = 0.0765$ m^4
Area of verticals	$A_v = 0.006$ m^2
Area of horizontals	$A_v = 0.004$ m^2
Area of diagonals	$A_v = 0.003$ m^2

In the first case with 32 degrees of freedom, the mass matrix is the identity matrix and the stiffness matrix is tridiagonal:

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & \cdot & 0 & 0 \\ -k_2 & k_2 + k_3 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & k_{n-1} + k_n & -k_n \\ 0 & 0 & \cdot & -k_n & k_n \end{bmatrix} \tag{63}$$

The sensitivity of the first five eigenvectors was calculated with respect to k_2 for all k equal to 1. Table 1 gives the number of flops and CPU time required for the methods as well as the relative error in percentages.

A weight equal to twice the eigenvalue was used to weigh the normality and orthogonality conditions in the iterative procedure.

The second example, shown in Fig. 1 and termed GARTEUR2,[13] has 78 beam elements with 72 nodes with three degrees of freedom per node for a total of 216 degrees of freedom.

The geometrical and material parameters for the structure are given in Table 2. Ten modes were used to calculate the sensitivity of the first 10 modes for one of the parameters highlighted in Fig. 1. A weight equal to the eigenvalue was used for this example. The results for this are given in Table 3.

The iterative least-squares method requires more CPU time and requires more flops than the modal expansion, whereas Nelson's technique falls somewhere in between. The modal expansion method would require more time for more precision particularly since additional modes would need to be included for greater accuracy. The added time and calculations of the iterative least-square technique are due primarily to the large number of iterations required. It has more accurate results, however, than the modal expansion method.

Table 3 GARTEUR2 (10 modes, $W = \omega^2$)

Method	Nelson	Iterative	Modal
flops	6.35*E*6	1.04*E*8	1.03*E*6
CPU(s)	4.76	36.75	0.12
method	least	squares	modal
mode	no. iter.	error, %	error, %
1	10	7.45*E*-8	1.82*E*-3
2	17	2.29*E*-4	2.77*E*-2
3	30	1.91*E*-2	6.69*E*-1
4	23	3.90*E*-2	4.74*E*-1
5	31	9.89*E*-2	3.20
6	94	1.60*E*-1	2.37
7	31	1.90*E*-1	3.84
8	51	1.24*E*-1	5.13
9	44	3.69*E*-1	33.90
10	131	2.78	73.26

Conclusions

In 1968, Fox and Kapoor[2] presented two techniques for calculating the sensitivity of eigenvectors, one iterative in nature and the other based on a modal expansion of the sensitivity. With regard to convergence of the modal expansion technique, they commented that the method "is being explored at this writing, but current results are inconclusive." In this paper, we show that the modal expansion scheme leads to a formulation for which the terms due to the higher modes can be larger than those from the lower modes, thus rendering it questionable in most applications since it could have considerable error. We present two criteria that can be used to assess the results obtained by the modal expansion technique.

More importantly, the iterative scheme developed by Fox and Kapoor,[2] when supplemented by orthogonality conditions incorporated by Rudisill and Chu,[6] which are appropriately weighted, is shown to converge. To assure convergence, we develop the expression for the minimum weight to be applied to the normality and orthogonality conditions. The weight of each condition should be at least equal to the difference between the eigenvalue whose eigenvector sensitivity is being calculated and the lowest eigenvalue.

Because of the form of the equations, a Householder transformation can be used to iteratively solve for the eigenvector sensitivity. The only large matrix that needs to be decomposed is the stiffness matrix for all eigenvector sensitivity calculations and for any number of eigenvalues or parameters for which sensitivity is needed. Should the sensitivity of only one eigenvector be needed, however, Nelson's method is still the best method since only one decomposition of an $n \times n$ band matrix is needed. For the two numerical examples tested here, Nelson's method outperformed the iterative least-squares method presented here regardless of the number of eigenvector sensitivities sought.

Appendix: Householder Transformation

From Eqs. (41) and (42) the proof of the Householder transformation[13] is relatively straightforward by showing that the matrix times its inverse equals the identity matrix:

$$
\begin{aligned}
&[K + UWU'][K^{-1} - K^{-1}U[W^{-1} + U'K^{-1}U]^{-1}U'K^{-1}] \\
&\quad = KK^{-1} + UWU'K^{-1} - U[W^{-1} + U'K^{-1}U]^{-1}U'K^{-1} \\
&\quad - UWU'K^{-1}U[W^{-1} + U'K^{-1}U]^{-1}U'K^{-1} = I \\
&\quad + UW[I - [W^{-1} + U'K^{-1}U][W^{-1} + U'K^{-1}U]^{-1}]U'K^{-1} \\
&\quad = I
\end{aligned} \tag{A1}
$$

The double Householder decomposition of Eq. (59) is given by

$$
\begin{aligned}
&[K + UW_1U' + VW_2V']^{-1} = [Q + VW_2V']^{-1} \\
&\quad = Q^{-1} - Q^{-1}V\left[W_2^{-1} + V'Q^{-1}V\right]^{-1}V'Q^{-1}
\end{aligned} \tag{A2}
$$

in which

$$
\begin{aligned}
&Q^{-1} = [K + UW_1U']^{-1} = K^{-1} - K^{-1}U \\
&\quad \times \left[W_1^{-1} + U'K^{-1}U\right]^{-1}U'K^{-1}
\end{aligned} \tag{A3}
$$

In the case where the normality constraint is separated from the orthogonality constraints, Eq. (61), these equations reduce further to one matrix Householder decomposition for the orthogonality constraints and one scalar decomposition for the normality constraint,

$$
\begin{aligned}
&[K + UWU' + vwv']^{-1} = [Q + vwv']^{-1} \\
&\quad = Q^{-1} - \frac{1}{[1/w + v'Q^{-1}v]}Q^{-1}vv'Q^{-1}
\end{aligned} \tag{A4}
$$

where Q^{-1} is defined in Eq. (A3).

Acknowledgments

The first author expresses his appreciation to the Universitée de Franche-Comté for the invitation to spend three months as a visiting professor in the fall of 1993, during which time this work was performed. Also, the authors wish to thank Fred Austin for suggesting the convergence proof of the iterative method.

References

[1]Wittrick, W. H., "Rates of Change of Eigenvalues, with Reference to Buckling and Vibration Problems," *Journal of the Royal Aeronautical Society*, Vol. 66, Sept. 1962, pp. 590, 591.

[2]Fox, R. L., and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[3]Plaut, R. H., and Huseyin, K., "Derivatives of Eigenvalues and Eigenvectors in Non-Self-Adjoint Systems," *AIAA Journal*, Vol. 11, No. 2, 1973, pp. 250, 251.

[4]Wang, B. P., "Improved Approximate Methods for Computing Eigenvector Derivatives in Structural Dynamics," *AIAA Journal*, Vol. 29, No. 6, 1991, pp. 1018–1020.

[5]Rudisill, C. S., "Derivatives of Eigenvalues and Eigenvectors for a General Matrix," *AIAA Journal*, Vol. 12, No. 5, 1974, pp. 721, 722.

[6]Rudisill, C. S., and Chu, Y.-Y., "Numerical Methods for Evaluating the Derivatives of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 13, No. 6, 1975, pp. 834–837.

[7]Andrew, A. L., "Iterative Computation of Derivatives of Eigenvalues and Eigenvectors," *Institute of Mathematics and Its Applications*, Vol. 24, Sept. 1979, pp. 209–218.

[8]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[9]Murthy, D., and Haftka, R. T., "Derivatives of Eigenvalues and Eigenvectors of a General Matrix," *International Journal for Numerical Methods in Engineering*, Vol. 26, 1988, pp. 293–311.

[10]Cardani, C., and Mantegazza, P., "Calculation of Eigenvalue and Eigenvector Derivatives for Algebraic Flutter and Divergence Eigenproblems," *AIAA Journal*, Vol. 17, No. 4, 1979, pp. 408–412.

[11]Ojalvo, I. U., "Efficient Computation of Mode-Shape Derivatives for Large Dynamic Systems," *AIAA Journal*, Vol. 25, No. 10, 1987, pp. 1386–1390.

[12]Householder, A. S., *Principles of Numerical Analysis*, McGraw–Hill, New York, 1953.

[13]Cogan, S., Ben-Haim, Y., and Ayer, F., "Modal Selective Sensitivity: Academic Examples," Workshop on Identification and Diagnosis of Mechanical Structures, Besançon, France, Nov. 1993.